Electromagnetic Fields

Vectors

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About the course

- Level: 2nd year Electrical Engineering
- Lectures: 3 hours weekly
- Tutorial: 2 hours
- Lab Oral Exam: None
- Total Marks: 125 (85 Final + 40 Term Work)

Objectives

- To develop understanding of the underplaying physics of electromagnetic phenomena.
- Be familiar with the formulation of the electromagnetic problem
- Utilize the different factors affecting the electromagnetic problems
- Manage the different materials and their properties.

Course Outline

- **Vectors**
- **Electrical Fields**
  - Electric field strength
  - Electrical flux density
  - Potential and energy
  - Dielectric and capacitance
- **Magnetic Fields**
  - Magnetic fields
  - Magnetic forces and torque
  - Inductance
  - Boundary Conditions
- **Time varying magnetic fields**

Type of Quantities

Quantity is a property that can exist as a magnitude or multitude.
Quantities can be compared in terms of ”more”, ”less”, or ”equal”, or by:
assigning a numerical value in terms of a unit of measurement.

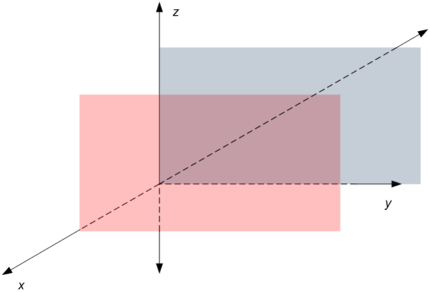
- Scale : has a value, but no direction
- Phasor : is a scale that depend on more than one variable.
- Vector : has a magnitude and direction

Free Space

Origin

- **Origin** is arbitrary point in free space considered as reference for all other points.
- A **coordinate** is a variable. Its constant value is geometrically represented by a **surface** in space.
- A **coordinate system** represents each point in the space by a unique vertex if the three surfaces of coordinates are perpendicular to each other.
- The origin has **zero** value in all coordinates.

Cartesian Coordinates System



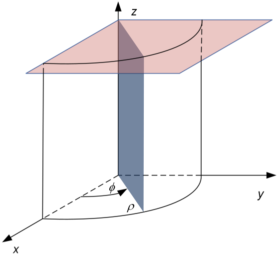
$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

- a constant value of x results a plan surface parallel to yz.
- a constant value of y results a plan surface parallel to zx.
- a constant value of z results a plan surface parallel to xy.

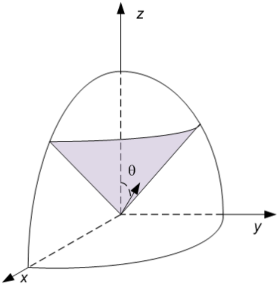
Cylindrical Coordinates System



$$0 < \rho < \infty$$
$$0 < \phi < 2\pi$$
$$-\infty < z < \infty$$

- a constant z value results a plan surface parallel to xy.
- a constant ρ value results a **cylindrical** surface around z.
- a constant ϕ value results a plan surface normal to both surfaces.

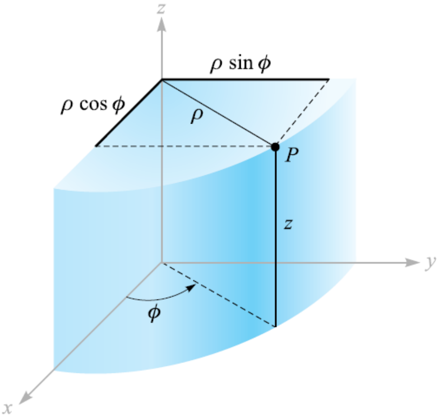
Spherical Coordinates



$$0 < r < \infty$$
$$0 < \theta < \pi$$
$$0 < \phi < 2\pi$$

- a constant r value results a spherical surface.
- a constant θ value results a **cone** surface around z with cone head at the origin.
- a constant ϕ value results a plan surface normal to both surfaces.

Coordinate Transformation



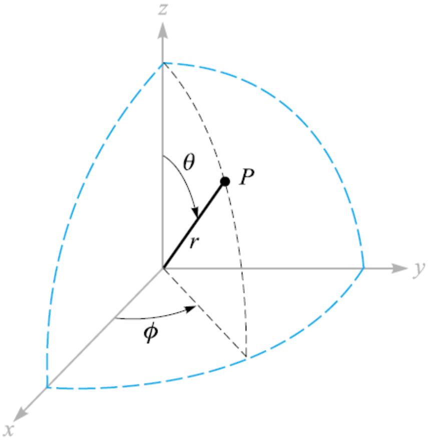
Cylindrical to Cartesian

$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$
$$z = z$$

Cartesian to Cylindrical

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

Coordinate Transformation



Spherical to Cartesian

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

Cartesian to Spherical

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$
$$\phi = \tan^{-1}(y/x)$$

Unit Vectors

A unit vector is a vector with magnitude of one unit in the vector direction. The unit vector usually referred to as $\overline{a}$.

The unit vectors of cartesian coordinates system are:

$$\overline{\mathbf{a_x}}, \overline{\mathbf{a_y}}, \overline{\mathbf{a_z}}$$

The unit vectors of cylindrical coordinates system are:

$$\overline{\mathbf{a}_\rho}, \overline{\mathbf{a}_\phi}, \overline{\mathbf{a_z}}$$

The unit vectors of spherical coordinates system are:

$$\overline{\mathbf{a_r}}, \overline{\mathbf{a}_\theta}, \overline{\mathbf{a}_\phi}$$

Point Representation

Origin:

$$O(0,0,0)$$

Arbitrary point:

$$P(x,y,z)$$

$$P(\rho\,\phi,z)$$

$$P(\,r,\theta,\phi)$$

$$P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A vector from Origin to pint P is determined by coordinate difference:

$$\overline{\mathbf{OP}} = P - O$$
$$\overline{\mathbf{OP}} = (x-0)\bar{\mathbf{a}}_{\mathbf{x}} + (y-0)\bar{\mathbf{a}}_{\mathbf{y}} + (z-0)\bar{\mathbf{a}}_{\mathbf{z}}$$
$$\overline{\mathbf{OP}} = x\bar{\mathbf{a}}_{\mathbf{x}} + y\bar{\mathbf{a}}_{\mathbf{y}} + z\bar{\mathbf{a}}_{\mathbf{z}}$$

Vector Magnitude and Unit

Vector magnitude of vector $\overline{\mathbf{A}}$, given by:

$$\overline{\mathbf{A}} = x\overline{\mathbf{a}}_{\mathbf{x}} + y\overline{\mathbf{a}}_{\mathbf{y}} + z\overline{\mathbf{a}}_{\mathbf{z}}$$

is given by:

$$|\overline{\mathbf{A}}| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector is defined as:

$$\overline{\mathbf{a}_{\mathbf{A}}} = \frac{\overline{\mathbf{A}}}{|\overline{\mathbf{A}}|}$$

$$\overline{\mathbf{A}} = |\overline{\mathbf{A}}|\overline{\mathbf{a}_{\mathbf{A}}}$$

$$\overline{\mathbf{A}} = \mathbf{A}\overline{\mathbf{a}_{\mathbf{A}}}$$

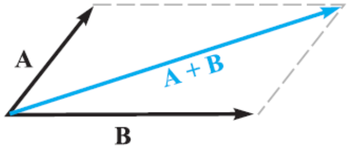
Vector Algebra

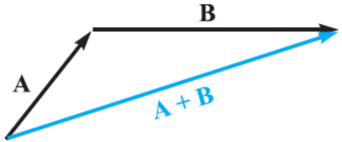
$$\overline{\mathbf{A}} = x_A \overline{\mathbf{a}}_\mathbf{x} + y_A \overline{\mathbf{a}}_\mathbf{y} + z_A \overline{\mathbf{a}}_\mathbf{z}$$

$$\overline{\mathbf{B}} = x_B \overline{\mathbf{a}}_\mathbf{x} + y_B \overline{\mathbf{a}}_\mathbf{y} + z_B \overline{\mathbf{a}}_\mathbf{z}$$

Addition:

$$\overline{\mathbf{A}} + \overline{\mathbf{B}} = \overline{\mathbf{B}} + \overline{\mathbf{A}} = (x_A + x_B)\overline{\mathbf{a}}_\mathbf{x} + (y_A + y_B)\overline{\mathbf{a}}_\mathbf{y} + (z_A + z_B)\overline{\mathbf{a}}_\mathbf{z}$$





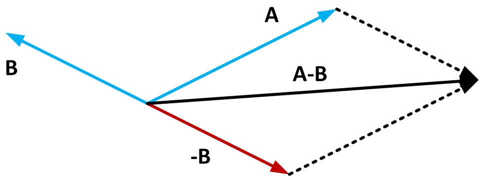
Vector Algebra

$$\overline{\mathbf{A}} = x_A \overline{\mathbf{a}}_\mathbf{x} + y_A \overline{\mathbf{a}}_\mathbf{y} + z_A \overline{\mathbf{a}}_\mathbf{z}$$

$$\overline{\mathbf{B}} = x_B \overline{\mathbf{a}}_\mathbf{x} + y_B \overline{\mathbf{a}}_\mathbf{y} + z_B \overline{\mathbf{a}}_\mathbf{z}$$

Substraction:

$$\overline{\mathbf{A}} - \overline{\mathbf{B}} = -(\overline{\mathbf{B}} - \overline{\mathbf{A}}) = (x_A - x_B)\overline{\mathbf{a}}_\mathbf{x} + (y_A - y_B)\overline{\mathbf{a}}_\mathbf{y} + (z_A - z_B)\overline{\mathbf{a}}_\mathbf{z}$$



Product (Scale)

$$\overline{\mathbf{A}} = x_A \overline{\mathbf{a}}_{\mathbf{x}} + y_A \overline{\mathbf{a}}_{\mathbf{y}} + z_A \overline{\mathbf{a}}_{\mathbf{z}}$$

If S is a scale:

$$S\overline{\mathbf{A}} = S\left(x_A \overline{\mathbf{a}}_{\mathbf{x}} + y_A \overline{\mathbf{a}}_{\mathbf{y}} + z_A \overline{\mathbf{a}}_{\mathbf{z}}\right)$$

$$S\overline{\mathbf{A}} = Sx_A \overline{\mathbf{a}}_{\mathbf{x}} + Sy_A \overline{\mathbf{a}}_{\mathbf{y}} + Sz_A \overline{\mathbf{a}}_{\mathbf{z}}$$

$$S\overline{\mathbf{A}} = S|A|\overline{\mathbf{a}_{\mathbf{A}}}$$

Product (Dot)

$$\overline{\mathbf{A}} = x_A\overline{\mathbf{a}}_{\mathbf{x}} + y_A\overline{\mathbf{a}}_{\mathbf{y}} + z_A\overline{\mathbf{a}}_{\mathbf{z}}$$

$$\overline{\mathbf{B}} = x_B\overline{\mathbf{a}}_{\mathbf{x}} + y_B\overline{\mathbf{a}}_{\mathbf{y}} + z_B\overline{\mathbf{a}}_{\mathbf{z}}$$

The dot product is defined as:

$$\overline{\mathbf{A}}.\overline{\mathbf{B}} = (x_A * x_B) + (y_A * y_B) + (z_A * z_B)$$

$$\overline{\mathbf{A}}.\overline{\mathbf{B}} = |\overline{\mathbf{A}}||\overline{\mathbf{B}}|.\cos\theta_{AB}$$

$$\overline{\mathbf{A}}.\overline{\mathbf{B}} = \overline{\mathbf{B}.\mathbf{A}}$$

$$\overline{\mathbf{A}}.(\overline{\mathbf{B}} + \overline{\mathbf{C}}) = \overline{\mathbf{A}}.\overline{\mathbf{B}} + \overline{\mathbf{A}}.\overline{\mathbf{C}}$$

Cross

$$\overline{\mathbf{A}} = x_A\overline{\mathbf{a_x}} + y_A\overline{\mathbf{a_y}} + z_A\overline{\mathbf{a_z}}$$

$$\overline{\mathbf{B}} = x_B\overline{\mathbf{a_x}} + y_B\overline{\mathbf{a_y}} + z_B\overline{\mathbf{a_z}}$$

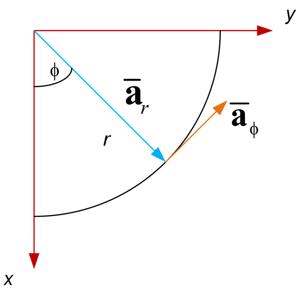
The cross product is defined as:

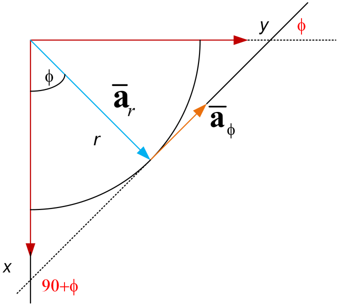
$$\overline{\mathbf{A}} \times \overline{\mathbf{B}} = |\overline{\mathbf{A}}||\overline{\mathbf{B}}|.\sin\theta_{AB}\overline{\mathbf{a_N}}$$

$$\overline{\mathbf{A}} \times \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{a_x}} & \overline{\mathbf{a_y}} & \overline{\mathbf{a_z}} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{bmatrix}$$

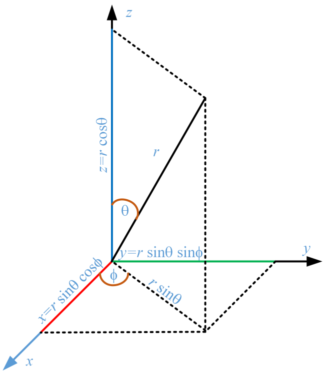
Vector Transformation

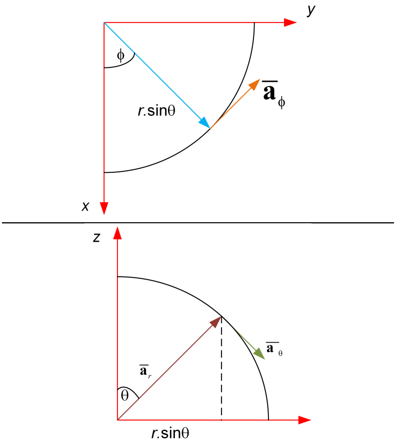
Cartesian and Cylindrical





	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	1





	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Differential Element
Cartesian

Length element

$$dl = dx\overline{\mathbf{a}}_\mathbf{x} + dy\overline{\mathbf{a}}_\mathbf{y} + dz\overline{\mathbf{a}}_\mathbf{z}$$

Area element

$$\overline{\mathbf{dA_x}} = dydz\overline{\mathbf{a_x}}$$
$$\overline{\mathbf{dA_y}} = dzdx\overline{\mathbf{a_y}}$$
$$\overline{\mathbf{dA_z}} = dxdy\overline{\mathbf{a_z}}$$

Volume Element

$$dV = dxdydz$$

Differential Element
Cylindrical

Length element

$$dl = d\rho\overline{\mathbf{a}}_\rho + \rho d\phi\overline{\mathbf{a}}_\phi + dz\overline{\mathbf{a}}_\mathbf{z}$$

Area element

$$\overline{\mathbf{dS_x}} = \rho d\phi dz\overline{\mathbf{a}_\rho}$$
$$\overline{\mathbf{dS_y}} = dzd\rho\overline{\mathbf{a}_\phi}$$
$$\overline{\mathbf{ds_z}} = \rho d\rho d\phi\overline{\mathbf{a}_\mathbf{z}}$$

Volume Element

$$dV = \rho d\rho d\phi dz$$

Differential Element
Spherical

Length element

$$dl = dr\overline{\mathbf{a}}_{\mathbf{r}} + rd\theta\overline{\mathbf{a}}_{\theta} + rsin\theta d\phi\overline{\mathbf{a}}_{\phi}$$

Area element

$$\overline{\mathbf{dS_r}} = r^2 sin\theta.d\theta d\phi\overline{\mathbf{a_r}}$$
$$\overline{\mathbf{dS}_\theta} = r\sin\theta.drd\theta\overline{\mathbf{a}_\theta}$$
$$\overline{\mathbf{dS}_\phi} = rdrd\theta\overline{\mathbf{a}_\phi}$$

Volume Element

$$dV = r^2 sin\theta drd\theta d\phi$$

Electromagnetic Fields

Vector‘s Calculus

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Scalar triple product

The scalar triple product (also called the mixed product, box product, or triple scalar product) is defined as the dot product of one of the vectors with the cross product of the other two.

$$\overline{\mathbf{A}}.(\overline{\mathbf{B}} \times \overline{\mathbf{C}})$$

is the (signed) volume of the parallelepiped defined by the three vectors given.

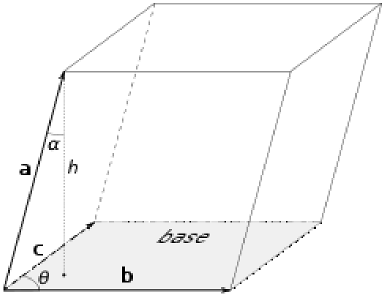
$$\overline{\mathbf{A}}.(\overline{\mathbf{B}} \times \overline{\mathbf{C}}) = \overline{\mathbf{B}}.(\overline{\mathbf{C}} \times \overline{\mathbf{A}}) = \overline{\mathbf{C}}.(\overline{\mathbf{A}} \times \overline{\mathbf{B}})$$

$$\overline{\mathbf{A}}.(\overline{\mathbf{B}} \times \overline{\mathbf{C}}) = -\overline{\mathbf{A}}.(\overline{\mathbf{C}} \times \overline{\mathbf{B}}) = -\overline{\mathbf{B}}.(\overline{\mathbf{A}} \times \overline{\mathbf{C}}) = -\overline{\mathbf{C}}.(\overline{\mathbf{B}} \times \overline{\mathbf{A}})$$

Scalar triple product

$$\overline{\mathbf{A}}.(\overline{\mathbf{B}} \times \overline{\mathbf{C}}) = \begin{bmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{bmatrix}$$

- If the scalar triple product is equal to zero, then the three vectors a, b, and c are coplanar.
- If any two vectors in the triple scalar product are equal, then its value is zero.



$$V = hS = (|\overline{\mathbf{A}}| \cos \alpha)(|\overline{\mathbf{B}}||\overline{\mathbf{C}}| \sin \theta)$$

Vector triple product

The vector triple product is defined as the cross product of one vector with the cross product of the other two.

$$\overline{\mathbf{A}} \times (\overline{\mathbf{B}} \times \overline{\mathbf{C}})$$

Lagrange's formula

$$\overline{\mathbf{A}} \times (\overline{\mathbf{B}} \times \overline{\mathbf{C}}) = \overline{\mathbf{B}}(\overline{\mathbf{A}}.\overline{\mathbf{C}}) - \overline{\mathbf{C}}(\overline{\mathbf{A}}.\overline{\mathbf{B}})$$

Jacobi identity

$$\overline{\mathbf{A}} \times (\overline{\mathbf{B}} \times \overline{\mathbf{C}}) + \overline{\mathbf{B}} \times (\overline{\mathbf{C}} \times \overline{\mathbf{A}}) + \overline{\mathbf{C}} \times (\overline{\mathbf{A}} \times \overline{\mathbf{B}}) = 0$$

__________-

$$(\overline{\mathbf{A}} \times \overline{\mathbf{B}}) \times \overline{\mathbf{C}} = \overline{\mathbf{A}} \times (\overline{\mathbf{B}} \times \overline{\mathbf{C}}) - \overline{\mathbf{B}} \times (\overline{\mathbf{A}} \times \overline{\mathbf{C}})$$

These formulas are very useful in simplifying vector calculations in physics.

The Differential Operator $\bigtriangledown$

$$\overline{\bigtriangledown} = \left(\frac{\partial}{\partial x}\overline{\mathbf{a_x}} + \frac{\partial}{\partial y}\overline{\mathbf{a_y}} + \frac{\partial}{\partial z}\overline{\mathbf{a_z}} \right)$$

Gradiance

The vector derivative of a scalar field is called the gradient, and it can be represented as:

$$\overline{\bigtriangledown} f = \left(\frac{\partial f}{\partial x} \overline{\mathbf{a_x}} + \frac{\partial f}{\partial y} \overline{\mathbf{a_y}} + \frac{\partial f}{\partial z} \overline{\mathbf{a_z}} \right)$$

It always points in the direction of greatest increase off, and it has a magnitude equal to the maximum rate of increase at the point just like a standard derivative.

Divergence

The divergence of a vector field is a scalar function that can be represented by the dot product between the operator $\overline{\bigtriangledown}$ and the vector.

$$\overline{\bigtriangledown}.\overline{\mathbf{A}} = \frac{\partial x_A}{\partial x} + \frac{\partial y_A}{\partial y} + \frac{\partial z_A}{\partial z}$$

The divergence is roughly a measure of a vector field's increase in the direction it points; but more accurately, it is a measure of that field's tendency to converge toward or repel from a point.

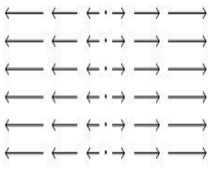
Curl

The curl of a vector field is a vector function that can be represented by the cross product of the operator $\overline{\bigtriangledown}$ and the vector field.

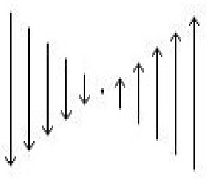
$$\overline{\bigtriangledown} \times \overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{a}}_x & \overline{\mathbf{a}}_y & \overline{\mathbf{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x_A & y_A & z_A \end{bmatrix}$$

The curl at a point is proportional to the on-axis torque to which a tiny pinwheel would be subjected if it were centered at that point.

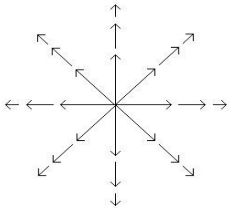
Examples



A vector field with positive divergence but zero curl.

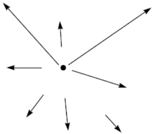


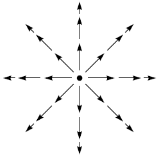
A vector field with nonzero curl (the curl vector points out of the computer screen), but zero divergence.

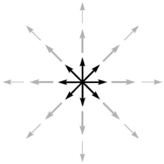


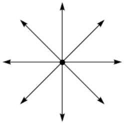
An inverse square field. Its divergence and curl are both zero

STREAMLINES AND SKETCHES OF FIELDS

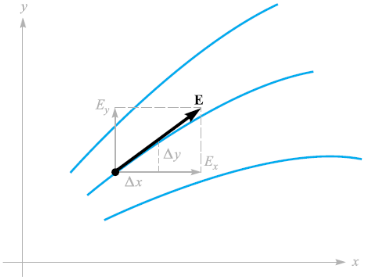

Poor sketch


fair sketch


fair sketch


streamline sketch

STREAMLINES AND SKETCHES OF FIELDS



$$\frac{E_y}{E_x} = \frac{dy}{dx}$$

A knowledge of the functional form of E_x and E_y (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

STREAMLINES Example

Consider the field:

$$\overline{\mathbf{E}} = \frac{1}{\rho}\overline{\mathbf{a}}_\rho$$

In rectangular coordinates:

$$\overline{\mathbf{E}} = \frac{x}{x^2+y^2}\overline{\mathbf{a}}_x + \frac{y}{x^2+y^2}\overline{\mathbf{a}}_y$$

Thus we form the differential equation:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x}$$

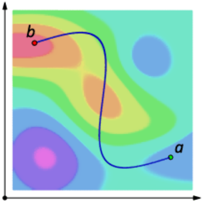
$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln y = \ln x + C$$

$$y = Cx$$

Line Integral

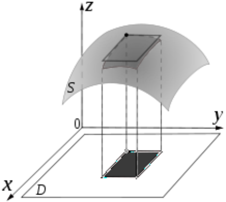
In mathematics, a line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used; contour integral as well, although that is typically reserved for line integrals in the complex plane.

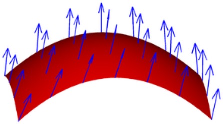


$$\oint_{path} f dl$$

Surface Integral

It can be thought of as the double integral analog of the line integral. Given a surface, one may integrate over its scalar fields (that is, functions which return scalars as values), and vector fields (that is, functions which return vectors as values).

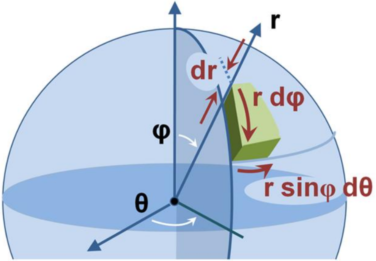




$$\oint_{surface} fds$$

Volume Integral

Volume integral refers to an integral over a 3-dimensional domain, that is, it is a special case of multiple integrals. Volume integrals are especially important in physics for many applications, for example, to calculate flux densities.



$$\oint_{volume} f dv$$

Gradient theorem

$$\int_a^b \overline{\bigtriangledown} T . d\bar{\mathbf{l}} = T(b) - T(a)$$

- The integration is independent of the path
- The integration over closed path equals zero

Divergence theorem (Gauss's theory, Green's Theory)

$$\int_{volume} (\overline{\bigtriangledown}.\overline{\mathbf{V}})dv = \oint_{surface} \overline{\mathbf{V}}.d\overline{\mathbf{s}}$$

The integral of the divergence over a volume equals to the value of the function at the boundary (the surface bounding the volume)

The Curl theory (Stokes' theorem)

$$\int_{surface} (\overline{\bigtriangledown} \times \overline{\mathbf{V}}).d\overline{\mathbf{S}} = \oint_{line} \overline{\mathbf{V}}.d\overline{\mathbf{l}}$$

- The integration of curl over a surface depends only on the boundary line, not on the particular surface used
- For any closed surface the integration equals zero

Electromagnetic Fields

Magnetic Materials

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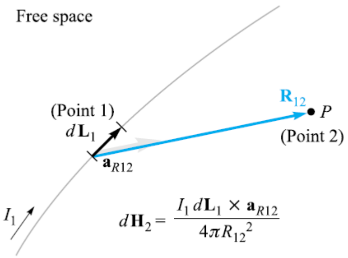
Introduction

- Definition of the magnetic field
- Show how magnetic field arises from a current distribution
- effect of this magnetic field on other currents
- Discus free space conditions, and the effect of material media

The source of the steady magnetic field

- a permanent magnet
- an electric field changing linearly with time
- a direct current.

Biot-Savart Law



$$d\overline{\mathbf{H}} = \frac{Id\overline{\mathbf{L}} \times \overline{\mathbf{a}}_R}{4\pi R^2} = \frac{Id\overline{\mathbf{L}} \times \overline{\mathbf{R}}}{4\pi R^3} \tag{1}$$

$$d\overline{\mathbf{H}}_2 = \frac{I_1 d\overline{\mathbf{L}}_1 \times \overline{\mathbf{a}}_{12}}{4\pi R_{12}^2} \tag{2}$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit that must be our experimental source, not the differential element. It follows that only the integral form of the **Biot-Savart law** can be verified experimentally,

$$\overline{\mathbf{H}} = \oint \frac{Id\overline{\mathbf{L}} \times \overline{\mathbf{a}}_R}{4\pi R^2}$$

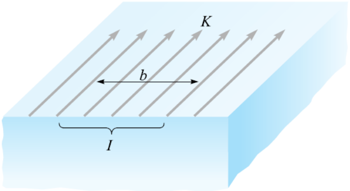
For uniform surface current density K:

$$I = Kb \tag{3}$$

For non-uniform surface current density K:

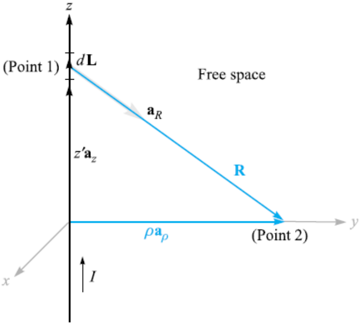
$$I = \int KdN \tag{4}$$

$$Id\overline{\mathbf{L}} = Kd\overline{\mathbf{s}} = \overline{\mathbf{J}}dv \tag{5}$$



$$\overline{\mathbf{H}} = \int_s \frac{K \times \overline{\mathbf{a}}_R d\overline{\mathbf{S}}}{4\pi R^2} \tag{6}$$

$$\overline{\mathbf{H}} = \int_{vol} \frac{\overline{\mathbf{J}} \times \overline{\mathbf{a}}_R dv}{4\pi R^2} \tag{7}$$

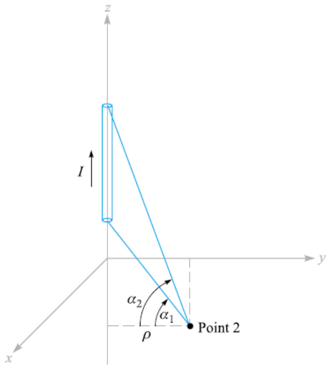


$$\overline{\mathbf{R}}_{12} = \overline{\mathbf{r}} - \overline{\mathbf{r}}' = \rho\overline{\mathbf{a}}_\rho - z'\overline{\mathbf{a}}_z$$

$$\overline{\mathbf{H}} = \int_{-\infty}^{\infty} \frac{I dz'\overline{\mathbf{a}}_z \times (\rho\overline{\mathbf{a}}_\rho - z'\overline{\mathbf{a}}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$\boxed{\overline{\mathbf{H}} = \frac{I}{2\pi\rho}\overline{\mathbf{a}}_\phi}$$

Field of a finite-length current element



$$\overline{\mathbf{H}} = \frac{I}{4\pi\rho}[\sin(\alpha_2) - \sin(\alpha_1)]\overline{\mathbf{a}}_\phi$$

Force on a moving charge

In an electric field:

$$\overline{\mathbf{F}} = Q\overline{\mathbf{E}} \tag{8}$$

In an magnetic field:

$$\overline{\mathbf{F}} = Q\overline{\mathbf{v}} \times \overline{\mathbf{B}} \tag{9}$$

A fundamental difference in the effect of the electric and magnetic fields on charged particles is now apparent, for a force which is always applied in a direction at right angles to the direction in which the particle is proceeding can never change the magnitude of the particle velocity.
In other words, the acceleration vector is always normal to the velocity vector. The kinetic energy of the particle remains unchanged, and it follows that the steady magnetic field is incapable of transferring energy to the moving charge.

Lorentz force equation

The force on a moving particle arising from combined electric and magnetic fields is obtained easily by superposition,

$$\overline{\mathbf{F}} = Q(\overline{\mathbf{E}} + \overline{\mathbf{v}} \times \overline{\mathbf{B}}) \tag{10}$$

Example

The point charge $Q = 18\mathrm{nC}$ has a velocity of $5 \times 10^6\ \mathrm{m/s}$ in the direction $\bar{\mathbf{a}}_v = 0.60\bar{\mathbf{a}}_\mathbf{x} + 0.75\bar{\mathbf{a}}_\mathbf{y} + 0.30\bar{\mathbf{a}}_\mathbf{z}$. Calculate the magnitude of the force exerted on the charge by the field:

1. $\bar{\mathbf{B}} = -3\bar{\mathbf{a}}_\mathbf{x} + 4\bar{\mathbf{a}}_\mathbf{y} + 6\bar{\mathbf{a}}_\mathbf{z}\ \mathrm{mT}$
2. $\bar{\mathbf{E}} = -3\bar{\mathbf{a}}_\mathbf{x} + 4\bar{\mathbf{a}}_\mathbf{y} + 6\bar{\mathbf{a}}_\mathbf{z}\ \mathrm{kV/m};$
3. $\bar{\mathbf{B}}$ and $\bar{\mathbf{E}}$ acting together.

Solution

$$\overline{\mathbf{F}}_m = Q|\overline{\mathbf{v}}|\overline{\mathbf{a}}_v \times \overline{\mathbf{B}}$$

$$\overline{\mathbf{F}}_m = 18 \times 10^{-9} \times 5 \times 10^{6} \begin{bmatrix} \overline{\mathbf{a}}_\mathbf{x} & \overline{\mathbf{a}}_\mathbf{y} & \overline{\mathbf{a}}_\mathbf{z} \\ 0.6 & 0.75 & 0.3 \\ -3 & 4 & 6 \end{bmatrix} \times 10^{-3}$$

$$\overline{\mathbf{F}}_m = 297\overline{\mathbf{a}}_\mathbf{x} - 405\overline{\mathbf{a}}_\mathbf{y} + 418.5\overline{\mathbf{a}}_\mathbf{z}\mu\mathrm{N}$$

$$|\overline{\mathbf{F}}_m| = 654\mu\mathrm{N}$$

Solution

$$\overline{\mathbf{F}}_e = Q\overline{\mathbf{E}}$$

The cross product is defined as:

$$\overline{\mathbf{F}}_e = 18 \times 10^{-9}(-3\overline{\mathbf{a}}_\mathbf{x} + 4\overline{\mathbf{a}}_\mathbf{y} + 6\overline{\mathbf{a}}_\mathbf{z}) \times 10^3$$

$$\overline{\mathbf{F}}_e = -54\overline{\mathbf{a}}_\mathbf{x} + 72\overline{\mathbf{a}}_\mathbf{y} + 108\overline{\mathbf{a}}_\mathbf{z}\mu\mathrm{N}$$

$$|\overline{\mathbf{F}}_e| = 140.6\mu\mathrm{N}$$

It should ne noted that $\overline{\mathbf{F}}_m > \overline{\mathbf{F}}_e$

$$\overline{\mathbf{F}} = 243\overline{\mathbf{a}}_\mathbf{x} + 333\overline{\mathbf{a}}_\mathbf{y} + 526.5\overline{\mathbf{a}}_\mathbf{z}\mu\mathrm{N}$$

Force on a Differential Current Element

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge as follows:

$$d\overline{\mathbf{F}} = dQ\overline{\mathbf{v}} \times \overline{\mathbf{B}} \tag{11}$$

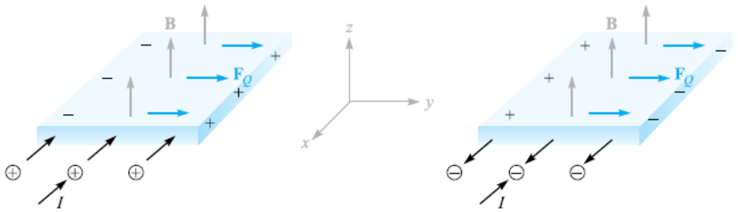
The differential force expressed by 11 is thus merely the sum of the forces on the individual charges. This sum, or resultant force, is not a force applied to a single object. In an analogous way, we might consider the differential gravitational force experienced by a small volume taken in a shower of falling sand.

Notes

- If our charges are electrons in motion in a conductor, however, we can show that the force is transferred to the conductor and that the sum of this extremely large number of extremely small forces is of practical importance.
- Within the conductor,electrons are in motion throughout a region of immobile positive ions which form a crystalline array, giving the conductor its solid properties.
- A magnetic field which exerts forces on the electrons tends to cause them to shift position slightly and produces a small displacement between the centers of (gravity) of the positive and negative charges.
- The Coulomb forces between electrons and positive ions, however, tend to resist such a displacement.

Hall Effect

- The charge separation that does result, however, is disclosed by the presence of a slight potential difference across the conductor sample in a direction perpendicular to both the magnetic field and the velocity of the charges.
- The voltage is known as the Hall voltage, and the effect itself is called the Hall effect.



Force on a moving charge

- Equal currents provided by holes and electrons in semiconductors can therefore be differentiated by their Hall voltages.
- This is one method of determining whether a given semiconductor is *n*-type or *p*-type.

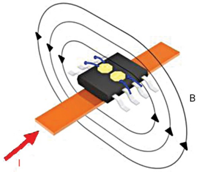
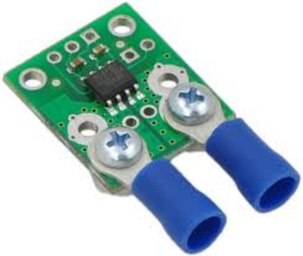
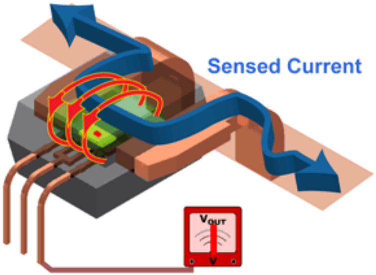


Figure: Current sensor [http://powerelectronics.com/]

Current Sensors





Forces on current-carrying conductors

Convection current density in terms of the velocity of the volume charge density:

$$\bar{\mathbf{J}} = \rho_v \bar{\mathbf{v}}$$

The differential element of charge in 11 may also be expressed in terms of volume charge density.

$$dQ = \rho_v dv$$

$$d\bar{\mathbf{F}} = dQ\bar{\mathbf{V}} \times \bar{\mathbf{B}}$$

$$d\bar{\mathbf{F}} = \rho_v dv(\frac{1}{\rho_v}\bar{\mathbf{J}}) \times \bar{\mathbf{B}}$$

$$d\bar{\mathbf{F}} = (\bar{\mathbf{J}} \times \bar{\mathbf{B}})dv \tag{12}$$

Forces on current-carrying conductors

$$\overline{\mathbf{J}}dv = \overline{\mathbf{K}}dS = Id\overline{\mathbf{L}}$$

Thus the Lorentz force equation may be applied to surface current density, or to a differential current filament,

$$d\overline{\mathbf{F}} = \overline{\mathbf{K}} \times \overline{\mathbf{B}}ds \tag{13}$$

$$d\overline{\mathbf{F}} = Id\overline{\mathbf{L}} \times \overline{\mathbf{B}} \tag{14}$$

Integrating the former equations respectively, leads to the integral formulations:

$$\overline{\mathbf{F}} = \int_{vol} \overline{\mathbf{J}} \times \overline{\mathbf{B}}dv \tag{15}$$

$$\overline{\mathbf{F}} = \int_{s} \overline{\mathbf{K}} \times \overline{\mathbf{B}}ds \tag{16}$$

$$\overline{\mathbf{F}} = \oint Id\overline{\mathbf{L}} \times \overline{\mathbf{B}} = -I\oint \overline{\mathbf{B}} \times d\overline{\mathbf{L}} \tag{17}$$

Force on straight conductor

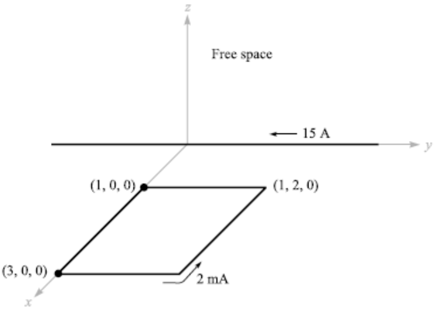
$$\overline{\mathbf{F}} = I\overline{\mathbf{L}} \times \overline{\mathbf{B}} \tag{18}$$

The magnitude of the force is given by the familiar equation:

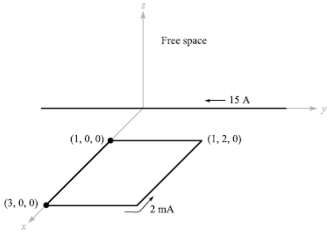
$$F = BIL\sin(\theta) \tag{19}$$

Example

The figure shows a loop of wire in the $z = 0$ plane carrying $2\mathrm{mA}$ in the field of an infinite filament on the y axis, as shown. Find, the total force on the loop.



Solution



$$\overline{\mathbf{F}}_{loop} = \sum_{i=1}^{4} \overline{\mathbf{F}}_{ci}$$

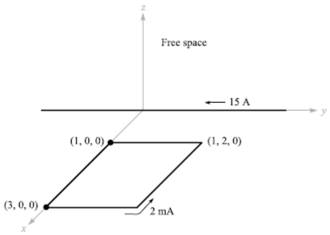
$$\overline{\mathbf{F}} = -I \oint \overline{\mathbf{B}} \times d\overline{\mathbf{L}}$$

$$\overline{\mathbf{F}} = -2 \times 10^{-3} \times 15 \times \frac{4\pi \times 10^{-7}}{2\pi}$$

$$\left(\int_{x=1}^{3} \frac{\overline{\mathbf{a}}_z}{x} \times dx\overline{\mathbf{a}}_x + \int_{y=0}^{2} \frac{\overline{\mathbf{a}}_z}{3} \times dy\overline{\mathbf{a}}_y \right.$$

$$\left. + \int_{x=3}^{1} \frac{\overline{\mathbf{a}}_z}{x} \times dx\overline{\mathbf{a}}_x + \int_{y=2}^{0} \frac{\overline{\mathbf{a}}_z}{1} \times dy\overline{\mathbf{a}}_y \right)$$

Solution



$$\overline{\mathbf{F}} = -6 \times 10^{-9} \times \left(\overline{\mathbf{a}}_y \ln(x)\Big|_1^3 - \overline{\mathbf{a}}_x y/3\Big|_0^2 + \overline{\mathbf{a}}_y \ln(x)\Big|_3^1 - \overline{\mathbf{a}}_x y\Big|_2^0 \right)$$

$$\overline{\mathbf{F}} = -6 \times 10^{-9} \times [(\frac{2}{3} - 0) + (0 - 2)](-\overline{\mathbf{a}}_x) = -8\overline{\mathbf{a}}_x \mathrm{nN}$$

FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS

It is possible to express the force on one current element directly in terms of a second current element without finding the magnetic field. Because we claimed that the magnetic-field concept simplifies our work, it then behooves us to show that avoidance of this intermediate step leads to more complicated expressions.

$$d\overline{\mathbf{H}}_2 = \frac{I_1 d\overline{\mathbf{L}}_1 \times \overline{\mathbf{a}}_{R12}}{4\pi R_{12}^2}$$

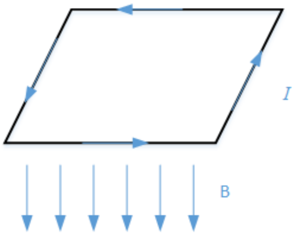
Now, the differential force on a differential current element is:

$$d\overline{\mathbf{F}} = Id\overline{\mathbf{L}} \times \overline{\mathbf{B}}$$

$$F_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\overline{\mathbf{L}}_2 \times \oint \frac{d\overline{\mathbf{L}}_1 \times \overline{\mathbf{a}}_{R_{12}}}{\overline{\mathbf{R}}_{12}^2} \right]$$

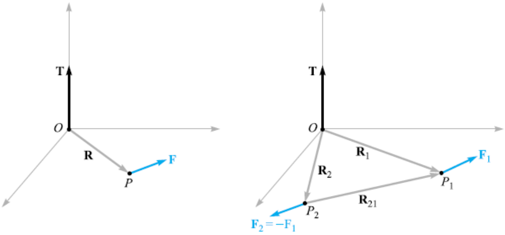
FORCE AND TORQUE ON A CLOSED CIRCUIT

Note: the force on a closed filamentary circuit in a uniform magnetic field is zero.



If the field is not uniform, the total force need not be zero.

FORCE AND TORQUE ON A CLOSED CIRCUIT



$$\overline{\mathbf{T}} = \overline{\mathbf{R}} \times \overline{\mathbf{F}}$$

$$\overline{\mathbf{T}} = \overline{\mathbf{R}}_1 \times \overline{\mathbf{F}}_1 + \overline{\mathbf{R}}_2 \times \overline{\mathbf{F}}_2$$

FORCE AND TORQUE

If the forces has equal magnitude and opposite direction:

$$\overline{\mathbf{F}}_1 = -\overline{\mathbf{F}}_2$$

$$\overline{\mathbf{F}}_1 + \overline{\mathbf{F}}_2 = 0$$

The the torque is:

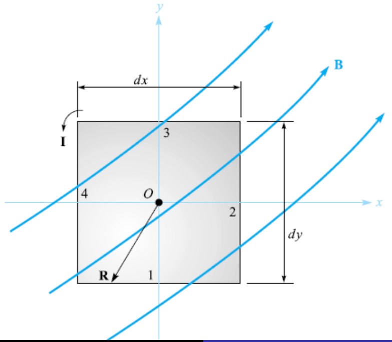
$$\overline{\mathbf{T}} = \overline{\mathbf{R}}_1 \times \overline{\mathbf{F}} - \overline{\mathbf{R}}_2 \times \overline{\mathbf{F}}$$

$$\overline{\mathbf{T}} = (\overline{\mathbf{R}}_1 - \overline{\mathbf{R}}_2) \times \overline{\mathbf{F}}$$

$$\overline{\mathbf{T}} = \overline{\mathbf{R}}_{21} \times \overline{\mathbf{F}}$$

Torque on a current loop

Now consider the torque on a differential current loop in a magnetic field $\overline{\mathbf{B}}$. The loop lies in the xy plane the sides of the loop are parallel to the x and y axes and are of length dx and dy. The value of the magnetic field at the center of the loop is taken as B_0.



Torque on a current loop

The vector force on side 1 is:

$$d\overline{\mathbf{F}}_1 = Idx\overline{\mathbf{a}}_x \times \overline{\mathbf{B}}_0$$

$$d\overline{\mathbf{F}}_1 = Idx(B_{0y}\overline{\mathbf{a}}_z - B_{0z}\overline{\mathbf{a}}_y)$$

$$\overline{\mathbf{R}} = -\frac{1}{2}dy\overline{\mathbf{a}}_y$$

$$d\overline{\mathbf{T}}_1 = \overline{\mathbf{R}}_1 \times d\overline{\mathbf{F}}_1$$

$$d\overline{\mathbf{T}}_1 = -\frac{1}{2}dxdyIB_{0y}\overline{\mathbf{a}}_x$$

$$d\overline{\mathbf{T}}_3 = -\frac{1}{2}dxdyIB_{0y}\overline{\mathbf{a}}_x$$

$$d\overline{\mathbf{T}}_1 + d\overline{\mathbf{T}}_3 = -dxdyIB_{0y}\overline{\mathbf{a}}_x$$

$$d\overline{\mathbf{T}}_2 + d\overline{\mathbf{T}}_4 = -dxdyIB_{0x}\overline{\mathbf{a}}_y$$

$$d\overline{\mathbf{T}} = Idxdy(B_{0x}\overline{\mathbf{a}}_y - B_{0y}\overline{\mathbf{a}}_x)$$

$$d\overline{\mathbf{T}} = Idxdy(\overline{\mathbf{a}}_z \times \overline{\mathbf{B}}_0)$$

$$d\overline{\mathbf{T}} = Id\overline{\mathbf{S}} \times \overline{\mathbf{B}} \tag{20}$$

We now define the product of the loop current and the vector area of the loop as the differential magnetic dipole moment *dm*, with units of Am^2. Thus:

$$d\overline{\mathbf{m}} = Id\overline{\mathbf{S}} \tag{21}$$

$$d\overline{\mathbf{T}} = d\overline{\mathbf{m}} \times \overline{\mathbf{B}} \tag{22}$$

Torque on a current loop

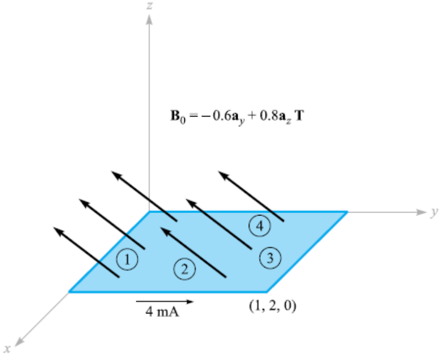
$$\overline{\mathbf{T}} = I\overline{\mathbf{S}} \times \overline{\mathbf{B}} = \overline{\mathbf{m}} \times \overline{\mathbf{B}} \tag{23}$$

We should note that the torque on the current loop always tends to turn the loop so as to align the magnetic field produced by the loop with the applied magnetic field that is causing the torque. This is perhaps the easiest way to determine the direction of the torque.

Example

Consider the rectangular loop shown. Calculate the torque by using the relation.

$$\overline{\mathbf{T}} = I\overline{\mathbf{S}} \times \overline{\mathbf{B}}$$



Solution

Consider the rectangular loop shown. Calculate the torque by using the relation.

$$\overline{\mathbf{T}} = I\overline{\mathbf{S}} \times \overline{\mathbf{B}}$$

$$\overline{\mathbf{T}} = (4 \times 10^{-3})(1 \times 2\overline{\mathbf{a}}_z) \times (0.0\overline{\mathbf{a}}_\mathbf{x} - 0.6\overline{\mathbf{a}}_\mathbf{y} + 0.8\overline{\mathbf{a}}_\mathbf{z})$$

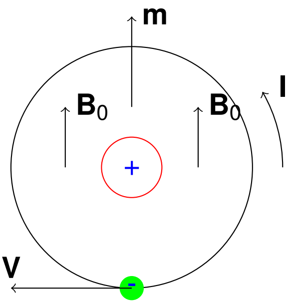
$$\overline{\mathbf{T}} = 4.8\overline{\mathbf{a}}_x \mathrm{mN\,m}$$

The Nature of Magnetic Materials

We are now in a position to combine our knowledge of the action of a magnetic field on a current loop with a simple model of an atom and obtain some appreciation of the difference in behavior of various types of materials in magnetic fields.

Although **accurate quantitative results can only be predicted through the use of quantum theory**, the simple atomic model, which assumes that there is a central positive nucleus surrounded by electrons in various circular orbits, yields reasonable quantitative results and provides a satisfactory qualitative theory.

Simple Atomic Model



- An electron in an orbit is analogous to a small current loop (in which the current is directed oppositely to the direction of electron travel).
- such, experiences a torque in an external magnetic field
- the torque tending to align the magnetic field produced by the orbiting electron with the external magnetic field

- If there were no other magnetic moments to consider, we would then conclude that all the orbiting electrons in the material would shift in such a way as to add their magnetic fields to the applied field.
- thus that the resultant magnetic field at any point in the material would be greater than it would be at that point if the material were not present.

Electron Spin

- It is necessary to digest the mathematics of relativistic quantum theory to show that an electron may have a spin magnetic moment of about $\pm 9 \times 10^{-24}$ Am2; the plus and minus signs indicate that alignment aiding or opposing an external magnetic field is possible.
- In an atom with many electrons present, only the spins of those electrons in shells which are not completely filled will contribute to a magnetic moment for the atom.

Nuclear Spin

- Although this factor provides a negligible effect on the overall magnetic properties of materials, it is the basis of the nuclear magnetic resonance imaging (MRI) procedure provided by many of the larger hospitals.

- Thus each atom contains many different component moments, and their combination determines the magnetic characteristics of the material and provides its general magnetic classification.

Types of Materials (Diamagnetic)

- Let us first consider atoms in which the small magnetic fields produced by the motion of the electrons in their orbits and those produced by the electron spin combine to produce a net field of zero.
- Note that we are considering here the fields produced by the electron motion itself in the absence of any external magnetic field;
- we might also describe this material as one in which the permanent magnetic moment m_0 of each atom is zero. Such a material is termed diamagnetic.
- It would seem, therefore, that an external magnetic field would produce no torque on the atom, no realignment of the dipole fields,
- Consequently, an internal magnetic field that is the same as the applied field.With an error that only amounts to

Paramagnetic

- When an external field is applied, however, there is a small torque on each atomic moment, and these moments tend to become aligned with the external field.
- This alignment acts to increase the value of $\overline{\mathbf{B}}$ within the material over the external value. However, the diamagnetic effect is still operating on the orbiting electrons and may counteract the increase.
- If the net result is a decrease in $\overline{\mathbf{B}}$, the material is still called diamagnetic.
- However, if there is an increase in $\overline{\mathbf{B}}$, the material is termed paramagnetic.

Characteristics of magnetic materials

Classificatio	Magnetic Moments	B Values	Comments
Diamagnetic	$\mathbf{m}_{\text{orb}} + \mathbf{m}_{\text{spin}} = 0$	$B_{\text{int}} < B_{\text{appl}}$	$B_{\text{int}} \doteq B_{\text{appl}}$
Paramagnetic	$\mathbf{m}_{\text{orb}} + \mathbf{m}_{\text{spin}} = \text{small}$	$B_{\text{int}} > B_{\text{appl}}$	$B_{\text{int}} \doteq B_{\text{appl}}$
Ferromagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} \gg B_{\text{appl}}$	Domains
Antiferromagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} \doteq B_{\text{appl}}$	Adjacent moments oppose
Ferrimagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} > B_{\text{appl}}$	Unequal adjacent moments oppose; low σ
Superparamagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} > B_{\text{appl}}$	Nonmagnetic matrix; recording tapes

MAGNETIZATION AND PERMEABILITY

- The current, will be the movement of bound charges (orbital electrons, electron spin, and nuclear spin), and the field, which has the dimensions of $\overline{\mathbf{H}}$, will be called the magnetization $\overline{\mathbf{M}}$.
- The current produced by the bound charges is called a bound current or Amperian current.

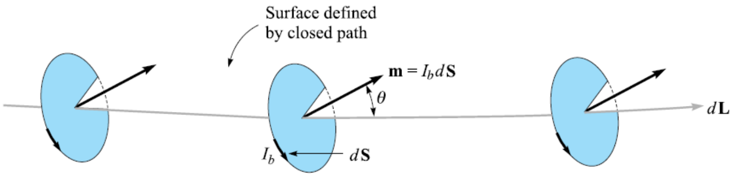
$$\overline{\mathbf{m}} = I_b d\overline{\mathbf{S}}$$

$$\overline{\mathbf{m}}_{total} = \sum_{i=1}^{n\Delta v} \overline{\mathbf{m}}_i \tag{24}$$

$$\overline{\mathbf{M}} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \overline{\mathbf{m}}_i$$

and see that its units must be the same as for $\overline{\mathbf{H}}$, amperes per meter.

MAGNETIZATION AND PERMEABILITY



Now let us consider the effect of some alignment of the magnetic dipoles as the result of the application of a magnetic field.

$$dI_B = nI_b d\overline{\mathbf{S}}.d\overline{\mathbf{L}} = \overline{\mathbf{M}}.d\overline{\mathbf{L}} \tag{25}$$

$$I_B = \oint \overline{\mathbf{M}}.d\overline{\mathbf{L}} \tag{26}$$

MAGNETIZATION AND PERMEABILITY

$$\oint \frac{\overline{\mathbf{B}}}{\mu_0} . d\overline{\mathbf{L}} = I_T \tag{27}$$

where

$$I_T = I_B + I$$

I is the total free current enclosed by the closed path. Note that the free current appears without subscript since it is the most important type of current and will be the only current appearing in Maxwell's equations.

$$I = I_T - I_B = \oint \left(\frac{\overline{\mathbf{B}}}{\mu_0} - \overline{\mathbf{M}} \right) . d\overline{\mathbf{L}} \tag{28}$$

MAGNETIZATION AND PERMEABILITY

$$\overline{\mathbf{H}} = \frac{\overline{\mathbf{B}}}{\mu_0} - \overline{\mathbf{M}} \tag{29}$$

That $\overline{\mathbf{B}} = \mu_0 \overline{\mathbf{H}}$ in free space where the magnetization is zero. This relationship is usually written in a form that avoids fractions and minus signs:

$$\overline{\mathbf{B}} = \mu_0 \left(\overline{\mathbf{H}} + \overline{\mathbf{M}} \right) \tag{30}$$

We may now use our newly defined H field in,

$$I = \oint \overline{\mathbf{H}} . d\overline{\mathbf{L}} \tag{31}$$

MAGNETIZATION AND PERMEABILITY

Using the several current densities, we have

$$I_B = \int_S \overline{\mathbf{J}}_B . d\overline{\mathbf{S}}$$

$$I_T = \int_S \overline{\mathbf{J}}_T . d\overline{\mathbf{S}}$$

$$I = \int_S \overline{\mathbf{J}} . d\overline{\mathbf{S}}$$

With the help of Stokes' theorem, we may therefore transform the above equation into the equivalent curl relationships:

$$\bigtriangledown \times \overline{\mathbf{M}} = \overline{\mathbf{J}}_B$$

$$\bigtriangledown \times \frac{\overline{\mathbf{B}}}{\mu_0} = \overline{\mathbf{J}}_T$$

$$\bigtriangledown \times \overline{\mathbf{H}} = \overline{\mathbf{J}}$$

MAGNETIZATION AND PERMEABILITY

The relationship between $\overline{\mathbf{B}}$, $\overline{\mathbf{H}}$, and $\overline{\mathbf{M}}$ may be simplified for linear isotropic media where a magnetic susceptibility χ_m can be defined:

$$\overline{\mathbf{M}} = \chi_m \overline{\mathbf{H}} \tag{32}$$

Then

$$\overline{\mathbf{B}} = \mu_0 \left(\overline{\mathbf{H}} + \chi \overline{\mathbf{H}}\right)$$
$$\overline{\mathbf{B}} = \mu_0 \mu_r \overline{\mathbf{H}}$$

Where, μ_r is defined as the relative permeability.

$$\mu_r = 1 + \chi_m \tag{33}$$

This enables us to write the simple relationship between $\overline{\mathbf{B}}$ and $\overline{\mathbf{H}}$,

$$\overline{\mathbf{B}} = \mu \overline{\mathbf{H}} \tag{34}$$

Example

Given a ferrite material that is operating in a linear mode with $\overline{\mathbf{B}} = 0.05\ \mathrm{T}$, $\mu_r = 50$, calculate values for χ_m , $\overline{\mathbf{M}}$, and $\overline{\mathbf{H}}$.

Answer

$$\mu_r = 1 + \chi_m$$

$$\chi_m = \mu_r - 1 = 49$$

$$B = \mu_r \mu_0 H$$

$$H = \frac{0.05}{50 \times 4\pi \times 10^{-7}} = 796 \mathrm{A\,m}$$

The magnetization is:

$$M = \chi_m H A/m.$$

PERMEABILITY

As,

$$\overline{\mathbf{B}} = \mu\overline{\mathbf{H}}$$

for homogeneous, linear, isotropic magnetic material that may be described in terms of a relative permeability μ_r

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \mu \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

For anisotropic materials

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Example

Find the magnetization in a magnetic material where:

- $\mu = 1.8 \times 10^{-5}$ H/m and $\overline{\mathbf{H}} = 120$ A/m;

Answer

- $$M = \chi_m H = (\mu_r - 1)H$$
  $$M = (\mu/\mu_0 - 1)H = (14.324 - 1) * 120$$
  $$M = 1598.873 \mathrm{A/m}$$

Example

Find the magnetization in a magnetic material where:

- $\mu_r = 22$, there are 8.3×10^{28} atoms/m^3, and each atom has a dipole moment of 4.5×10^{-27} A.m^2

Answer

- $$M = \texttt{atoms}/\mathrm{m}^3 * \texttt{dipole moment/atom}$$
  $$M = 8.3 \times 10^{28} \times 4.5 \times 10^{-27} = 373.5\mathrm{A}/\mathrm{m}$$

Example

Find the magnetization in a magnetic material where:

- $\overline{\mathbf{B}} = 300\mu\mathrm{T}$ and $\chi_m = 15$.

Answer

- $$B = \mu_0(H + M)$$
  $$B = \mu_0(1 + \chi_m)H$$
  $$B = \mu_0\left(\frac{B}{\mu_0(1 + \chi_m)} + M\right)$$
  $$M = \frac{B}{\mu_0} - \frac{B}{\mu_0(1 + \chi_m)}$$
  $$M = \frac{B}{\mu_0}\left(1 - \frac{1}{1 + \chi_m}\right)$$
  $$M = \frac{B}{\mu_0}\left(\frac{\chi_m}{1 + \chi_m}\right) = 223.81/\mathrm{m}$$

MAGNETIC BOUNDARY CONDITIONS

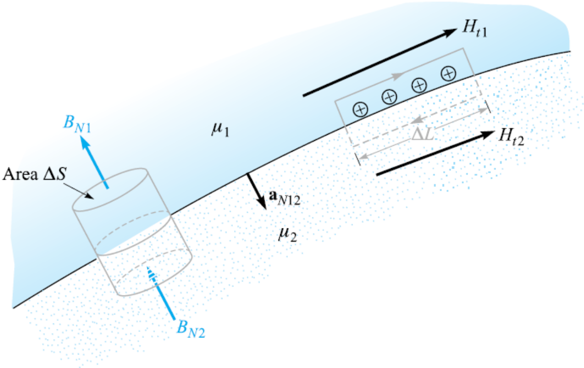


Figure: Boundary between two isotropic homogeneous linear materials with permeabilities μ_1 and μ_2

Allowing the surface to cut a small cylindrical gaussian surface. Applying Gauss's law for the magnetic field

$$\oint_S \overline{\mathbf{B}}.d\overline{\mathbf{S}} = 0$$

It is found that:

$$B_{N1}\Delta S - B_{N2}\Delta S = 0$$

$$B_{N2} = B_{N1} \quad (35)$$

Such:

$$H_{N2} = \frac{\mu_1}{\mu_2} H_{N2} \quad (36)$$

The relationship between the normal components of $\overline{\mathbf{M}}$, is found as:

$$M_{N2} = \chi_{m2}\frac{\mu_1}{\mu_2} H_{N1} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1}$$

From, Ampere's circuital law:

$$\int \overline{\mathbf{H}}.d\overline{\mathbf{L}} = I$$

Applying about a small closed path in a plane normal to the boundary surface. Taking a clockwise trip around the path, we find that

$$H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

where we assume that the boundary may carry a surface current K whose component normal to the plane of the closed path is K. Thus

$$H_{t1} - H_{t2} = K \tag{37}$$

The directions are specified more exactly by using the cross product to identify the tangential components as:

$$(H_1 - H_2) \times \overline{\mathbf{a}}_{N_{12}} = \overline{\mathbf{K}} \tag{38}$$

$$H_{t1} - H_{t2} = \overline{\mathbf{a}}_{N_{12}} \times \overline{\mathbf{K}}$$

For tangential $\overline{\mathbf{B}}$, we have:

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K$$

The boundary condition on the tangential component of the magnetization for linear materials is therefore:

$$M_{t2} = \frac{\chi_2}{\chi_1} M_{t1} - \chi_{m2} K$$

The last three boundary conditions on the tangential components are much simpler, of course, if the surface current density is zero. This is a free current density, and it must be zero if neither material is a conductor.

Example

Assume that $\mu = \mu 1 = 4\mu$H/m in region 1 where $z > 0$, whereas $\mu_2 = 7\mu$H/m in region 2 wherever $z < 0$. Moreover,let $\overline{\mathbf{K}} = 80\overline{\mathbf{a}}_x$ A/m on the surface $z = 0$. It is established a field, $\overline{\mathbf{B}}_1 = 2\overline{\mathbf{a}}_\mathbf{x} - 3\overline{\mathbf{a}}_\mathbf{y} + \overline{\mathbf{a}}_\mathbf{z}$ mT, in region 1 and seek the value of $\overline{\mathbf{B}}_2$.

Answer For the problem shown:

$$\overline{\mathbf{B}}_{N1} = \overline{\mathbf{a}}_z$$

$$\overline{\mathbf{B}}_{t1} = \overline{\mathbf{B}} - \overline{\mathbf{B}}_{N1} = 2\overline{\mathbf{a}}_x - 3\overline{\mathbf{a}}_y$$

$$\overline{\mathbf{B}}_{N2} = \overline{\mathbf{B}}_{N1} = \overline{\mathbf{a}}_z\mathrm{mT}$$

Example

Assume that $\mu = \mu 1 = 4\mu$H/m in region 1 where $z > 0$, whereas $\mu_2 = 7\mu$H/m in region 2 wherever $z < 0$. Moreover,let $\overline{\mathbf{K}} = 80\overline{\mathbf{a}}_x$ A/m on the surface $z = 0$. It is established a field, $\overline{\mathbf{B}}_1 = 2\overline{\mathbf{a}}_\mathbf{x} - 3\overline{\mathbf{a}}_\mathbf{y} + \overline{\mathbf{a}}_\mathbf{z}\ \mathrm{mT}$, in region 1 and seek the value of $\overline{\mathbf{B}}_2$.

Answer For the problem shown:

$$(H_{t1} - H_{t2}) = \overline{\mathbf{a}}_{N_{12}} \times \overline{\mathbf{K}}$$

$$\left(\frac{\overline{\mathbf{B}}_{t1}}{\mu_1} - \frac{\overline{\mathbf{B}}_{t2}}{\mu_2}\right) = \overline{\mathbf{a}}_{N_{12}} \times \overline{\mathbf{K}}$$

$$\overline{\mathbf{B}}_{t2} = \mu_2 \left(\frac{\overline{\mathbf{B}}_{t1}}{\mu_1} - \overline{\mathbf{a}}_{N_{12}} \times \overline{\mathbf{K}}\right)$$

$$\overline{\mathbf{B}}_{t2} = \mu_2 \left(\frac{\overline{\mathbf{B}}_{t1}}{\mu_1} - \overline{\mathbf{a}}_{N_{12}} \times \overline{\mathbf{K}}\right)$$

Example

Assume that $\mu = \mu 1 = 4\mu$H/m in region 1 where $z > 0$, whereas $\mu_2 = 7\mu$H/m in region 2 wherever $z < 0$. Moreover,let $\overline{\mathbf{K}} = 80\overline{\mathbf{a}}_x$ A/m on the surface $z = 0$. It is established a field, $\overline{\mathbf{B}}_1 = 2\overline{\mathbf{a}}_\mathbf{x} - 3\overline{\mathbf{a}}_\mathbf{y} + \overline{\mathbf{a}}_\mathbf{z}$ mT, in region 1 and seek the value of $\overline{\mathbf{B}}_2$.

Answer For the problem shown:

$$\overline{\mathbf{B}}_{t2} = \mu_2 \left(\frac{\overline{\mathbf{B}}_{t1}}{\mu_1} - \overline{\mathbf{a}}_{N_{12}} \times \overline{\mathbf{K}} \right)$$

$$\overline{\mathbf{B}}_{t2} = 7 \times 10^{-6} \left(\frac{2\overline{\mathbf{a}}_x - 3\overline{\mathbf{a}}_y}{4 \times 10^{-3}} - (-\overline{\mathbf{a}}_z) \times (80\overline{\mathbf{a}}_x) \right)$$

$$\overline{\mathbf{B}}_{t2} = 3.5\overline{\mathbf{a}}_\mathbf{x} - 4.69\overline{\mathbf{a}}_\mathbf{y} + 0\overline{\mathbf{a}}_\mathbf{z}\mathrm{mT}$$

$$\overline{\mathbf{B}}_2 = \overline{\mathbf{B}}_{N2} + \overline{\mathbf{B}}_{t2} = 3.5\overline{\mathbf{a}}_\mathbf{x} - 4.69\overline{\mathbf{a}}_\mathbf{y} + \overline{\mathbf{a}}_\mathbf{z}$$

The Magnetic Circuit

In this section, we digress briefly to discuss the fundamental techniques involved in solving a class of magnetic problems known as magnetic circuits. As we will see shortly, the name arises from the great similarity to the dc-resistive-circuit analysis with which it is assumed we are all familiar.

$$\overline{\mathbf{E}} = - \bigtriangledown V \tag{39}$$

$$\overline{\mathbf{H}} = - \bigtriangledown V_m \tag{40}$$

$$V_{AB} = \int_A^B \overline{\mathbf{E}}.d\overline{\mathbf{L}} \tag{41}$$

$$V_{m_{AB}} = \int_A^B \overline{\mathbf{H}}.d\overline{\mathbf{L}} \tag{42}$$

$$\overline{\mathbf{J}} = \sigma \overline{\mathbf{E}} \tag{43}$$

$$\overline{\mathbf{B}} = \mu \overline{\mathbf{H}} \tag{44}$$

$$I = \int \overline{\mathbf{J}}.d\overline{\mathbf{S}} \tag{45}$$

$$\Phi = \int \overline{\mathbf{B}}.d\overline{\mathbf{S}} \tag{46}$$

$$V = IR \tag{47}$$

$$V_m = \Phi \Re \tag{48}$$

$$R = \frac{d}{\sigma S} \tag{49}$$

$$\oint \overline{\mathbf{E}}.d\overline{\mathbf{L}} = 0 \tag{51}$$

$$\oint \bar{\mathbf{I}}.d\overline{\mathbf{L}} = I_{total} \tag{52}$$

$$\oint \bar{\mathbf{I}}.d\overline{\mathbf{L}} = NI \tag{53}$$

POTENTIAL ENERGY AND FORCES

$$W_E = \frac{1}{2}\int_{vol} \overline{\mathbf{D}}.\overline{\mathbf{E}}dv \tag{54}$$

$$W_H = \frac{1}{2}\int_{vol} \mu H^2 dv \tag{55}$$

$$W_H = \frac{1}{2}\int_{vol} \frac{B^2}{\mu} dv \tag{56}$$

$$\tag{57}$$

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POTENTIAL ENERGY AND FORCES

We were able to find an expression for the energy in an electrostatic field by establishing the work necessary to bring the prerequisite point charges from infinity to their final resting places. The general expression for energy is

$$W_E = \frac{1}{2}\int_{vol} \overline{\mathbf{D}}.\overline{\mathbf{E}}dv \tag{1}$$

The total energy stored in a steady magnetic field in which $\overline{\mathbf{B}}$ is linearly related to $\overline{\mathbf{H}}$ is

$$W_H = \frac{1}{2}\int_{vol} \overline{\mathbf{B}}.\overline{\mathbf{H}}dv \tag{2}$$

$$W_H = \frac{1}{2}\int_{vol} \mu H^2 dv = \frac{1}{2}\int_{vol} \frac{B^2}{\mu} dv \tag{3}$$

POTENTIAL ENERGY AND FORCES ON MAGNETIC MATERIALS

The expression for the energy in an electrostatic field is given by:

$$W_E = \frac{1}{2}\int_{vol} \overline{\mathbf{D}}.\overline{\mathbf{E}} dv$$

INDUCTANCE

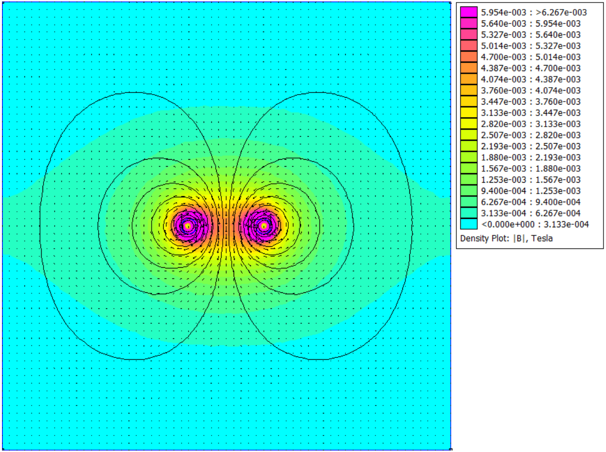
- Inductance is the last of the three familiar parameters from circuit theory that we are defining in more general terms.
- As a prelude to defining inductance, we first need to introduce the concept of flux linkage.
- Let us consider a toroid of N turns in which a current I produces a total flux ϕ.
- We assume first that this flux links or encircles each of the N turns, and we also see that each of the N turns links the total flux ϕ. The flux linkage $N\phi$ is defined as the product of the number of turns N and the flux ϕ linking each of them.

Inductance

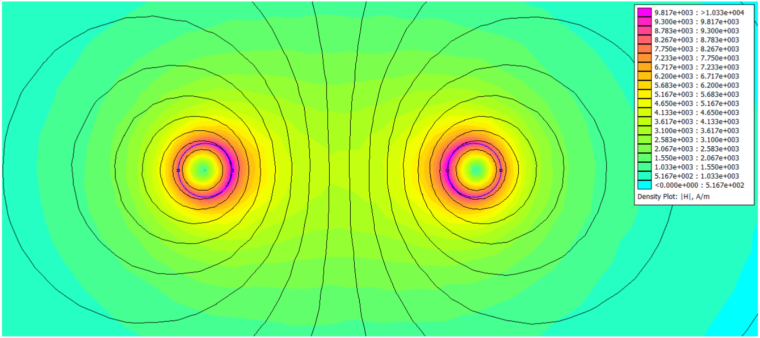
We now define inductance (or self-inductance) as the ratio of the total flux linkages to the current which they link:

$$L = \frac{N\phi}{I} = \frac{\lambda}{I}$$

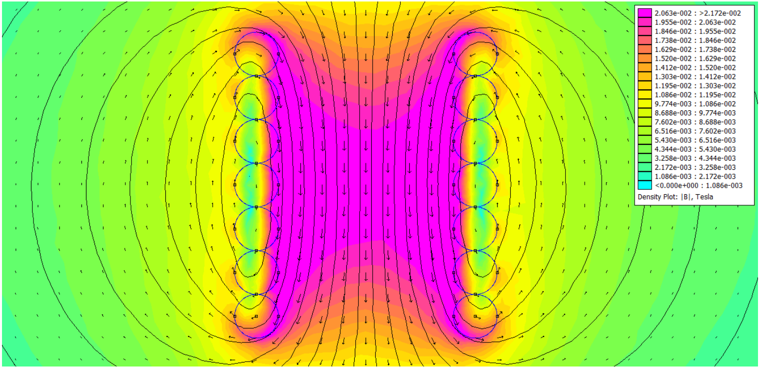
Flux linkage of one turn



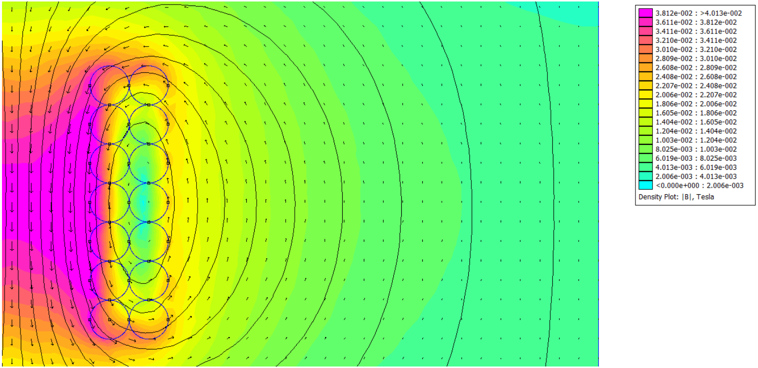
Flux linkage of one turn



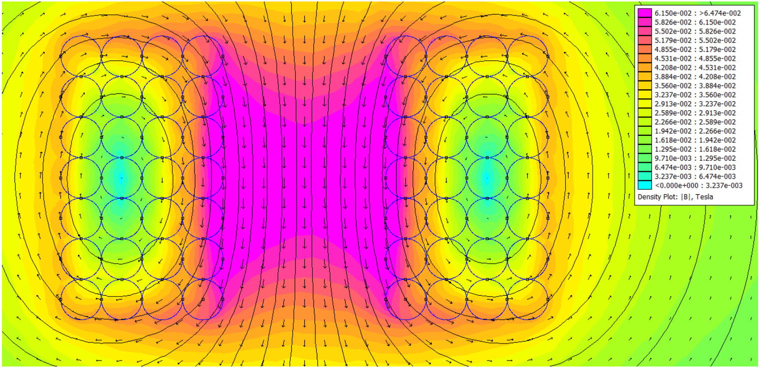
Flux linkage of multi turn coil



Flux linkage of multi turn coil



Flux linkage of multi turn coil



Inductance of coaxial cable

For $(a < \rho < b)$:

$$H_\phi = \frac{I}{2\pi\rho} \tag{4}$$

$$\overline{\mathbf{B}} = \mu_0 \overline{\mathbf{H}} = \frac{\mu_0 I}{2\pi\rho}\overline{\mathbf{a}}_\phi \tag{5}$$

$$\Phi = \int_S \overline{\mathbf{B}}.d\overline{\mathbf{S}} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\overline{\mathbf{a}}_\phi.d\rho dz\overline{\mathbf{a}}_\phi \tag{6}$$

$$\Phi = \frac{\mu_0 Id}{2\pi} \ln \frac{b}{a} \tag{7}$$

Inductance per meter length

$$L = \frac{N\phi}{I} = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \tag{8}$$

In order to obtain the total flux linkages we must look at the coil on a turn-by-turn basis.

$$N\Phi = \Phi_1 + \Phi_2 + \Phi_3 + ... + \Phi_N$$
$$N\Phi = \sum_{i=0}^{N} = \Phi_i \tag{9}$$

where ϕ_i is the flux linking the i^{th} turn. Rather than doing this, we usually rely on experience and empirical quantities called winding factors and pitch factors to adjust the basic formula to apply to the real physical world.

Mutual Inductance

We conclude by defining the mutual inductance between two circuits 1 and 2, M12, in terms of mutual flux linkages,

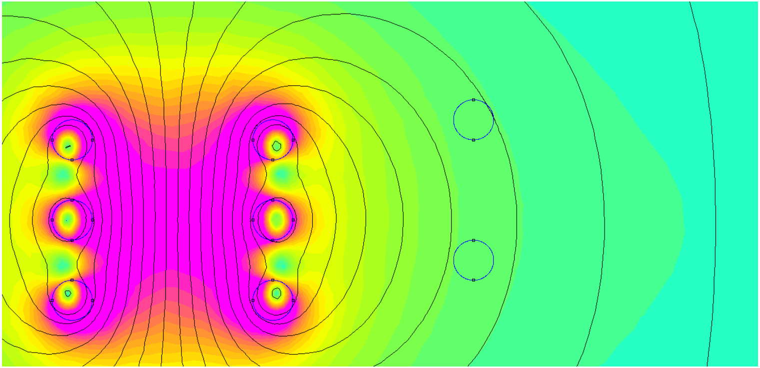
$$M_{12} = \frac{N_2 \Phi 12}{I_1}$$

Interchange of the subscripts does not change the right-hand side and therefore:

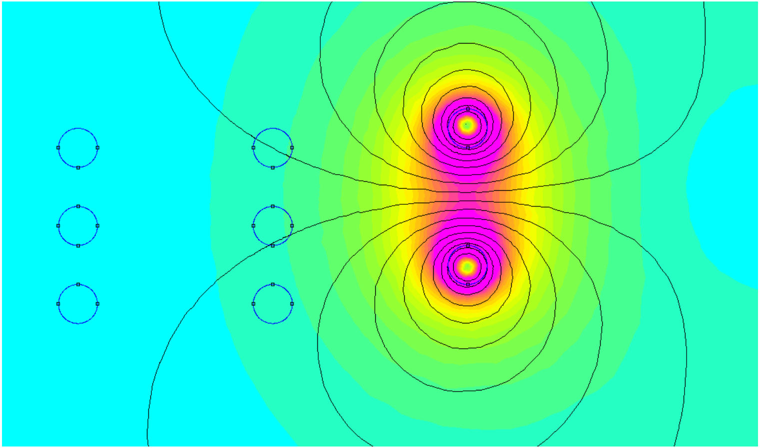
$$M_{12} = M_{21}$$

Mutual inductance is also measured in henrys, and we rely on the context to allow us to differentiate it from magnetization, also represented by M.

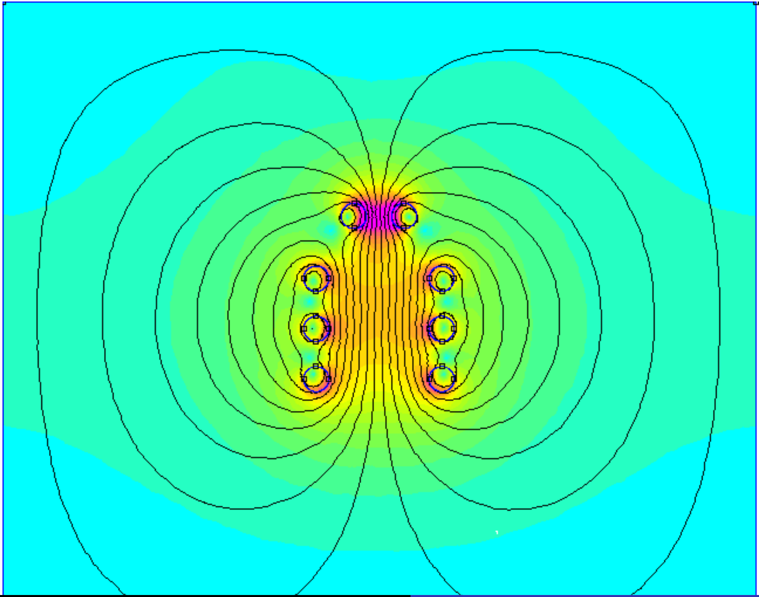
Mutual Inductance angle 90°



Mutual Inductance angle 90°



Mutual Inductance angle 0°



Time Varying Fields

After Oersted demonstrated in 1820 that an electric current affected a compass needle, Faraday professed his belief that if a current could produce a magnetic field, then a magnetic field should be able to produce a current.

The concept of the field was not available at that time, and Faraday's goal was to show that a current could be produced by magnetism.

Time Varying Fields

In terms of fields, we now say that a time-varying magnetic field produces an electromotive force (emf) that may establish a current in a suitable closed circuit. Faraday's law is customarily stated as:

$$emf = -N\frac{d\Phi}{dt}$$

A nonzero value of $d\Phi/dt$ may result from any of the following situations:

1. A time-changing flux linking a stationary closed path
2. Relative motion between a steady flux and a closed path
3. A combination of the two

The statement that the induced voltage acts to produce an opposing flux is known as Lenz's law.

Time Varying Fields (Transformer Voltage)

$$emf = \oint \overline{\mathbf{E}}.dL \tag{10}$$

$$emf = \oint \overline{\mathbf{E}}.dL = -\frac{d}{dt}\int_S \overline{\mathbf{B}}.d\overline{\mathbf{S}} \tag{11}$$

$$emf = \oint \overline{\mathbf{E}}.dL = -\int_S \frac{\partial \overline{\mathbf{B}}}{\partial t}.d\overline{\mathbf{S}} \tag{12}$$

from Stoke's theorem to the closed line integral:

$$\int_S (\overline{\nabla} \times \overline{\overline{\mathbf{E}}}).d\overline{\mathbf{S}} = -\int_S \frac{\partial \overline{\mathbf{B}}}{\partial t}.d\overline{\mathbf{S}}$$

$$\overline{\nabla} \times \overline{\overline{\mathbf{E}}} = -\frac{\partial \overline{\mathbf{B}}}{\partial t} \tag{13}$$

Time Varying Fields (Motional Voltage)

$$\overline{\mathbf{F}} = Q\overline{\mathbf{v}} \times \overline{\mathbf{B}} \tag{14}$$

$$\frac{\overline{\mathbf{F}}}{Q} = \overline{\mathbf{v}} \times \overline{\mathbf{B}} \tag{15}$$

Motional electric filed intensity:

$$\overline{\mathbf{E}}_m = \overline{\mathbf{v}} \times \overline{\mathbf{B}} \tag{16}$$

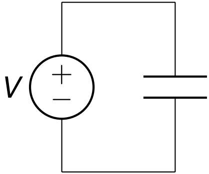
Motional emf

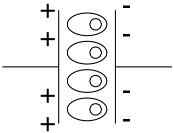
$$emf = \oint \overline{\mathbf{E}}.d\overline{\mathbf{L}} = \oint (\overline{\mathbf{v}} \times \overline{\mathbf{B}}).d\overline{\mathbf{L}} \tag{17}$$

Kirchhoff voltage law

$$emf = \oint \overline{\mathbf{E}}.d\overline{\mathbf{L}} = -\int_S \frac{\partial \overline{\mathbf{B}}}{\partial t} d\overline{\mathbf{S}} + \oint (\overline{\mathbf{v}} \times \overline{\mathbf{B}}).d\overline{\mathbf{L}} \tag{18}$$

Displacement Current





$$V = \int \overline{\mathbf{E}}.d\overline{\mathbf{L}} = Ed \tag{19}$$

$$C = \frac{\epsilon A}{d} \tag{20}$$

$$I = C\frac{dV}{dt} = JA = \frac{\epsilon A}{d}\frac{d}{dt}Ed \tag{21}$$

$$\overline{\mathbf{J}} = \epsilon\frac{d\overline{\mathbf{E}}}{dt} = \frac{d\overline{\mathbf{D}}}{dt} \tag{22}$$

Displacement Current

$$\overline{\nabla} \times \overline{\mathbf{H}} = \overline{\mathbf{J}} + \frac{d\overline{\mathbf{D}}}{dt} \tag{23}$$

$$\overline{\nabla} \times \overline{\mathbf{H}} = \overline{\mathbf{J}} + \overline{\mathbf{J}}_d \tag{24}$$

$$\oint \overline{\mathbf{H}}.d\overline{\mathbf{L}} = I + I_d = I + \int_S \frac{d\overline{\mathbf{D}}}{dt}.d\overline{\mathbf{S}} \tag{25}$$

Maxwell's Equations

The differential form

$$\overline{\nabla} \times \overline{\mathbf{E}} = -\frac{\partial \overline{\mathbf{B}}}{\partial t} \tag{26}$$

$$\overline{\nabla} \times \overline{\mathbf{H}} = \overline{\mathbf{J}} + \frac{\partial \overline{\mathbf{D}}}{\partial t} \tag{27}$$

$$\overline{\nabla}.\overline{\mathbf{D}} = \rho_v \tag{28}$$

$$\overline{\nabla}.\overline{\mathbf{B}} = 0 \tag{29}$$

Auxiliary equations

$$\overline{\mathbf{D}} = \epsilon\overline{\mathbf{E}} \qquad \overline{\mathbf{D}} = \epsilon_0\overline{\mathbf{E}} + \overline{\mathbf{P}} \qquad \overline{\mathbf{P}} = \chi_e\epsilon_0\overline{\mathbf{E}}$$

$$\overline{\mathbf{B}} = \mu\overline{\mathbf{H}} \qquad \overline{\mathbf{B}} = \mu_0(\overline{\mathbf{H}} + \overline{\mathbf{M}}) \qquad \overline{\mathbf{M}} = \chi_m\overline{\mathbf{H}}$$

$$\overline{\mathbf{J}} = \sigma\overline{\mathbf{E}} \qquad \overline{\mathbf{J}} = \rho_v\overline{\mathbf{v}}$$

Maxwell's Equations

Integral form:

$$\oint \overline{\mathbf{E}}.d\overline{\mathbf{L}} = -\int_S \frac{\partial \overline{\mathbf{B}}}{\partial t}.d\overline{\mathbf{S}}$$

$$\oint \overline{\mathbf{H}}.d\overline{\mathbf{L}} = I + \int_S \frac{\partial \overline{\mathbf{D}}}{\partial t}.d\overline{\mathbf{S}}$$

$$\oint_S \overline{\mathbf{D}}.d\overline{\mathbf{S}} = \int_{vol} \rho_v dv$$

$$\oint_S \overline{\mathbf{B}}.d\overline{\mathbf{S}} = 0$$